B.A./B.Sc. I (SEMESTER-II) - L.U.

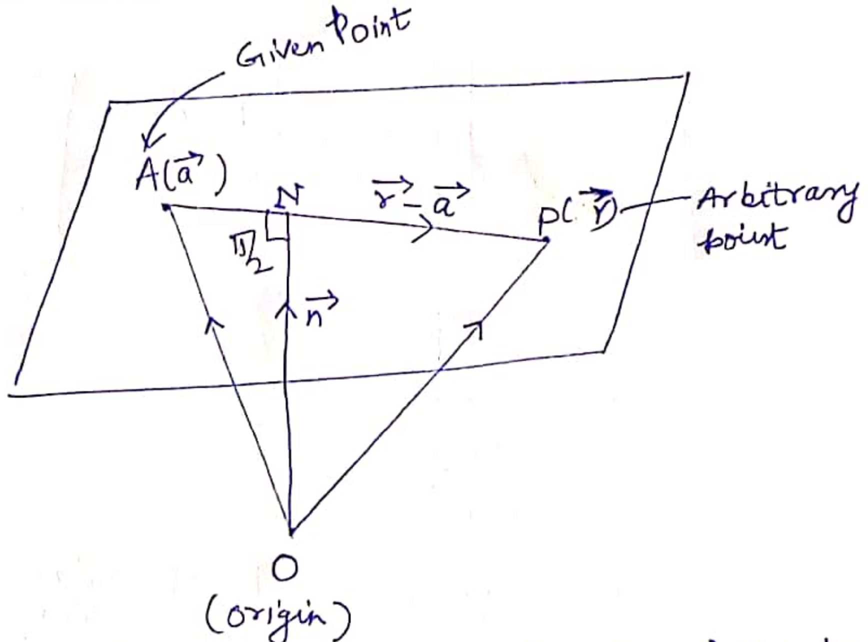
MATHEMATICS - PAPER-II (GEOMETRY)

UNIT-II (PLANE - VECTOR APPROACH)

(E-CONTENT / LECTURE NOTES-2)

(PREPARED BY: Dr. DEEPAK KUMAR SRIVASTAVA)

§ Plane Passes Through a Given Point and Perpendicular to a Given Vector.



Let $A(\vec{a})$ be the given point whose position vector with respect to origin is $\vec{a}$ and $\overrightarrow{ON} = \vec{n}$.

Let $\vec{r}$ be the position vector of arbitrary point P which lies on the plane passing through given point A and perpendicular to ON.

Then $\overrightarrow{AP} = \vec{r} - \vec{a}$ must be perpendicular to $\overrightarrow{ON} = \vec{n}$.

Therefore, by applying perpendicularity condition, we have

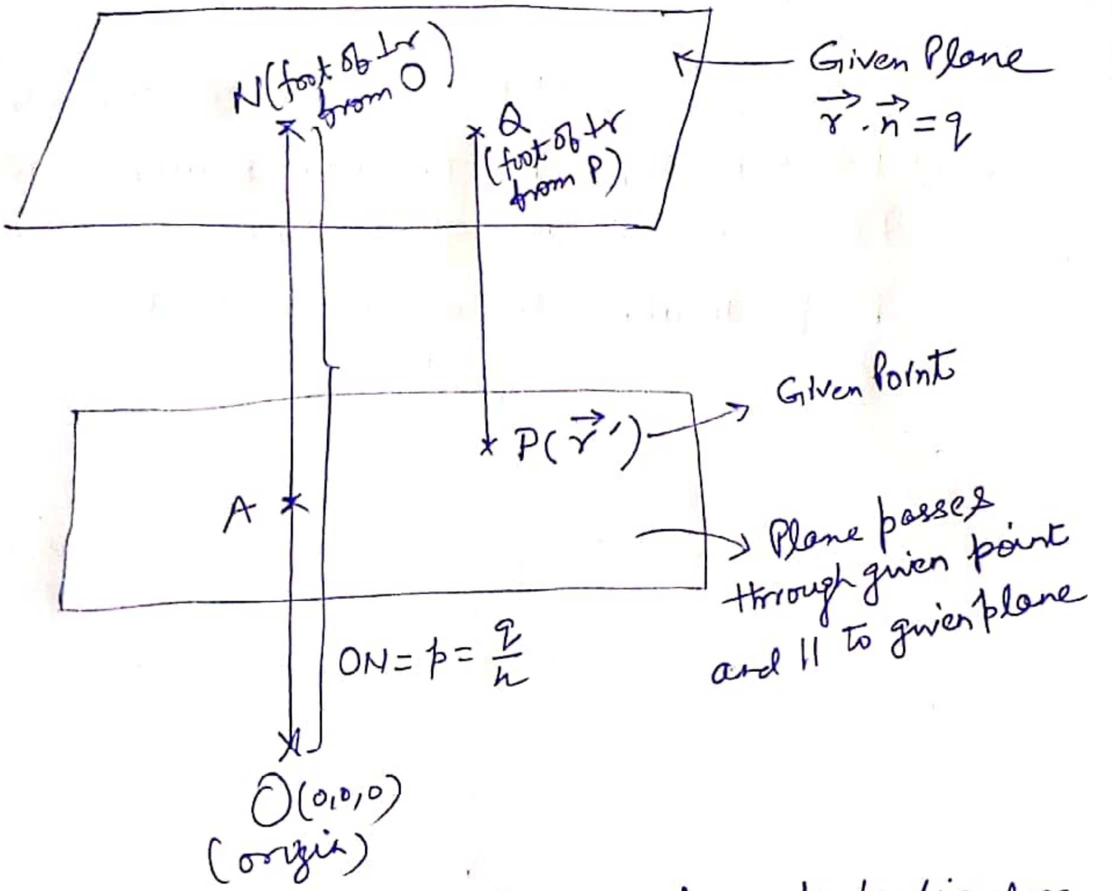
$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

or

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

which is the required vector equation of plane.

§ Perpendicular Distance of a Point from a Plane



Let $P(\vec{r}\,')$ be any given point whose perpendicular distance from plane $\vec{r}\cdot\vec{n}=q$ ① is to be evaluated.

The equation of plane through point $P(\vec{r}\,')$ and perpendicular to ON is given by $\vec{r}\cdot\vec{n}=\vec{r}\,'\cdot\vec{n}$ ....②

The perpendicular distance $OA = \dfrac{\vec{r}\,'\cdot\vec{n}}{n}$, $n=|\vec{n}|$ ...③

Now, the required perpendicular distance is the distance between two parallel planes ① & ②, i.e;

$$= ON - OA = \frac{q}{n} - \frac{\vec{r}\,'\cdot\vec{n}}{n}$$

$$= \frac{q-\vec{r}\,'\cdot\vec{n}}{n}.$$

§ Two Sides of a Plane

Theorem. Two points whose position vectors are $\vec{a}$ and $\vec{b}$ are on the same or opposite side of a plane $\vec{r}\cdot\vec{n}=q$ according as $\vec{a}\cdot\vec{n}-q$ and $\vec{b}\cdot\vec{n}-q$ are of the same or opposite signs.

Proof. Let $A(\vec{a})$ and $B(\vec{b})$ are two given points and the equation of plane is $\vec{r}\cdot\vec{n}=q$ —① Suppose that the line segment AB cut the plane ① in point P where P divides AB in ratio $m:l$ i.e; $AP:PB=m:l$. Then the position vector of P is $\dfrac{m\vec{b}+l\vec{a}}{m+l}$.

Since the point P lies over plane ①, then we must have

$$\left(\frac{m\vec{b}+l\vec{a}}{m+l}\right)\cdot\vec{n}=q \text{ or } (m\vec{b}+l\vec{a})\cdot\vec{n}=(m+l)q$$

$$\Rightarrow m(\vec{b}\cdot\vec{n}-q)=-l(\vec{a}\cdot\vec{n}-q)$$

$$\Rightarrow \frac{m}{l}=-\left(\frac{\vec{a}\cdot\vec{n}-q}{\vec{b}\cdot\vec{n}-q}\right) \quad \text{—— ②}$$

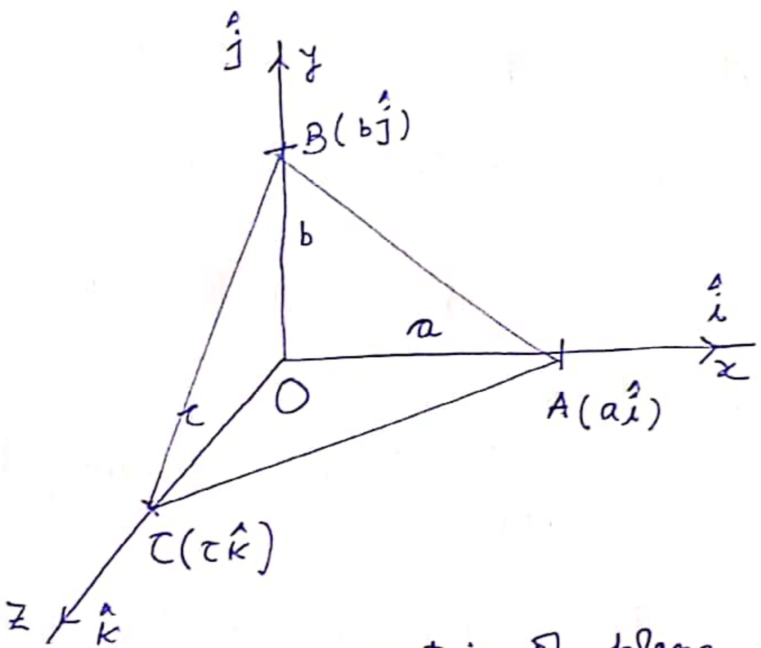
Now, we discuss following two cases —

CASE-1 If the ratio $\frac{m}{l}$ is +ve, then the expressions $\vec{a}\cdot\vec{n}-q$ and $\vec{b}\cdot\vec{n}-q$ are of opposite signs and P divides AB internally. Hence, A and B must be on the opposite sides of the plane.

<u>CASE-2</u> If the ratio $\frac{m}{l}$ is −ve, then the expressions $\vec{a}\cdot\vec{n}-q$ and $\vec{b}\cdot\vec{n}-q$ are of same signs and point P divides AB externally. Hence, A & B are on the same side of plane.

<u>Hence Proved.</u>

§ <u>INTERCEPTING FORM</u> & <u>INTERCEPTS</u>



Let us consider the equation of plane is $\vec{r}\cdot\vec{n}=q$ where $\vec{n}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and a, b, c are the intercepts made by plane on coordinate axes. The intersecting points on coordinate axes are $A(a\hat{i})$, $B(b\hat{j})$, $C(c\hat{k})$ respectively.

On satisfying the position vectors of these intersecting points in equation of plane ①, we get

$a\hat{i}\cdot\vec{n} = q \quad \Rightarrow a\hat{i}\cdot(a_1\hat{i}+b_1\hat{j}+c_1\hat{k}) = q \Rightarrow a_1 = \frac{q}{a}$

$b\hat{j}\cdot\vec{n} = q \quad \Rightarrow b\hat{j}\cdot(a_1\hat{i}+b_1\hat{j}+c_1\hat{k}) = q \Rightarrow b_1 = \frac{q}{b}$

$c\hat{k}\cdot\vec{n} = q \quad \Rightarrow c\hat{k}\cdot(a_1\hat{i}+b_1\hat{j}+c_1\hat{k}) = q \Rightarrow c_1 = \frac{q}{c}$

Also, $a = \dfrac{q}{\hat{i}\cdot\vec{n}}$, $b = \dfrac{q}{\hat{j}\cdot\vec{n}}$, $c = \dfrac{q}{\hat{k}\cdot\vec{n}}$.

Now, the equation of plane $\vec{r}\cdot\vec{n} = q$ becomes

$$\vec{r}\cdot\left(\frac{q}{a}\hat{i} + \frac{q}{b}\hat{j} + \frac{q}{c}\hat{k}\right) = q$$

$$\Rightarrow \boxed{\vec{r}\cdot\left(\frac{1}{a}\hat{i} + \frac{1}{b}\hat{j} + \frac{1}{c}\hat{k}\right) = 1} \qquad (\because q \neq 0)$$

This vector equation is known as the intercepting form of plane with intercepts a, b, c made by plane on coordinate axes respectively.

SOLVED EXAMPLES BASED ON LECTURE NOTES-2

EXAMPLE-1 The equation of plane passing through a point $\hat{i}+2\hat{j}+3\hat{k}$ and perpendicular to vector $2\hat{i}-3\hat{j}+5\hat{k}$ is:

(a) $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k}) = 9$

(b) $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k}) = 10$

(c) $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k}) = 11$

(d) $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k}) = 12$

Solution. We know that, the equation of plane passing through a point $\vec{a}$ and perpendicular to vector $\vec{n}$ is $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$.

Here, $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{n} = 2\hat{i}-3\hat{j}+5\hat{k}$,

then the equation of plane is

$$\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k}) = (\hat{i}+2\hat{j}+3\hat{k})\cdot(2\hat{i}-3\hat{j}+5\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k}) = 11.$$

So, the answer is (c).

EXAMPLE-2 The perpendicular distance from a point $\hat{i}+\hat{j}+2\hat{k}$ onto the plane $\vec{r}\cdot(2\hat{i}+2\hat{j}+\hat{k})=15$ is:

(a) 3 units

(b) 4 units

(c) 5 units

(d) 6 units

Solution. We know that the perpendicular distance from a point $\vec{r}'$ onto the plane $\vec{r}\cdot\vec{n} = q$ is $\dfrac{q - \vec{r}'\cdot\vec{n}}{n}$ where $n = |\vec{n}|$.

Given that, $\vec{n} = 2\hat{i}+2\hat{j}+\hat{k}$, $n = |\vec{n}| = \sqrt{4+4+1}$
$= \sqrt{9} = 3$.

$q = 15$, $\vec{r}' = \hat{i}+\hat{j}+2\hat{k}$

Therefore, the required perpendicular distance

$$d = \frac{q - \vec{r}'\cdot\vec{n}}{n} = \frac{15 - (\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}+2\hat{j}+\hat{k})}{3}$$

$$= \frac{15-6}{3} = \frac{9}{3} = 3 \text{ units}$$

So, the answer is (a).

EXAMPLE-3 The location of two points $\hat{i}+\hat{j}-2\hat{k}$ and $2\hat{i}+3\hat{j}-\hat{k}$ with respect to plane $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 9$ is:

(a) one lies on the plane and other outside the plane

(b) lies on the plane

(c) lies on opposite side

(d) lies on same side

Solution. We know that two points $\vec{a}$ and $\vec{b}$ lies on same or opposite side with respect to the plane $\vec{r}\cdot\vec{n} = q$ when signs of $\vec{a}\cdot\vec{n} - q$ & $\vec{b}\cdot\vec{n} - q$ are same or opposite.

Here, $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$, $q = 9$

consider $\vec{a} \cdot \vec{n} - q = (\hat{i} + \hat{j} - 2\hat{k}) \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 9 = -12$

and $\vec{b} \cdot \vec{n} - q = (2\hat{i} + 3\hat{j} - \hat{k}) \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 9 = -4$

$\therefore$ the signs of $\vec{a} \cdot \vec{n} - q$ & $\vec{b} \cdot \vec{n} - q$ are same.

So, the answer is (d).

EXAMPLE-4 The intercepting form of plane $\vec{r} \cdot (2\hat{i} + 2\hat{j} + \hat{k}) = 8$ is:

(a) $\vec{r} \cdot (\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{4}\hat{k}) = 1$

(b) $\vec{r} \cdot (\frac{1}{4}\hat{i} + \frac{1}{4}\hat{j} + \frac{1}{8}\hat{k}) = 1$

(c) $\vec{r} \cdot (\frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k}) = 1$

(d) $\vec{r} \cdot (\frac{1}{5}\hat{i} + \frac{1}{5}\hat{j} + \frac{1}{10}\hat{k}) = 1$

Solution. We know that $\vec{r} \cdot (\frac{1}{a}\hat{i} + \frac{1}{b}\hat{j} + \frac{1}{c}\hat{k}) = 1$ is the intercepting form of plane $\vec{r} \cdot (a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) = q$ where $a_1 = \frac{q}{a}$, $b_1 = \frac{q}{b}$, $c_1 = \frac{q}{c}$ with intercept a, b, c on coordinate axes. Also,

$$a = \frac{q}{\hat{i} \cdot \vec{n}}, \quad b = \frac{q}{\hat{j} \cdot \vec{n}}, \quad c = \frac{q}{\hat{k} \cdot \vec{n}}.$$

Now, consider the equation of plane

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} + \hat{k}) = 8$$

divide the equation with scalar 8 on both sides, we get $\vec{r} \cdot (\frac{1}{4}\hat{i} + \frac{1}{4}\hat{j} + \frac{1}{8}\hat{k}) = 1$.

So, the answer is (b).

EXAMPLE-5 The intercept made by plane $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=9$ on the coordinate axes are:

(a) $1, \frac{3}{2}, 3$

(b) $4, \frac{5}{2}, 1$

(c) $3, \frac{1}{2}, 5$

(d) $9, \frac{9}{2}, 3$

Solution. The intercept made by plane $\vec{r}\cdot\vec{n}=q$ onto the coordinate axes are;

$$a = \frac{q}{\hat{i}\cdot\vec{n}}, \quad b = \frac{q}{\hat{j}\cdot\vec{n}}, \quad c = \frac{q}{\hat{k}\cdot\vec{n}}$$

so, the intercepts are

$$a = \frac{9}{\hat{i}\cdot(\hat{i}+2\hat{j}+3\hat{k})} = \frac{9}{1} = 9$$

$$b = \frac{9}{\hat{j}\cdot(\hat{i}+2\hat{j}+3\hat{k})} = \frac{9}{2}$$

$$c = \frac{9}{\hat{k}\cdot(\hat{i}+2\hat{j}+3\hat{k})} = \frac{9}{3} = 3$$

$\therefore$ the answer is (d).

ALITOR. Consider the equation of plane

$$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 9 \quad \text{——— (1)}$$

divide the equation by scalar 9 on both sides,

we get $$\vec{r}\cdot\left(\frac{1}{9}\hat{i}+\frac{1}{9/2}\hat{j}+\frac{1}{3}\hat{k}\right)=1.$$

The intercept made by plane (1) on coordinate axes are: $9, 9/2, 3$.

EXERCISE BASED ON LECTURE NOTES-2

Q1. Equation of plane passes through a point $2\hat{i}+3\hat{j}-5\hat{k}$ and perpendicular to vector $3\hat{i}+5\hat{j}-\hat{k}$ is:

(a) $\vec{r}\cdot(3\hat{i}+5\hat{j}-\hat{k}) = 25$

(b) $\vec{r}\cdot(3\hat{i}+5\hat{j}-\hat{k}) = 26$

(c) $\vec{r}\cdot(3\hat{i}+5\hat{j}-\hat{k}) = 27$

(d) $\vec{r}\cdot(3\hat{i}+5\hat{j}-\hat{k}) = 28$

(Ans: (b))

Q2. The perpendicular distance of point $2\hat{i}+3\hat{j}-7\hat{k}$ from plane $\vec{r}\cdot(2\hat{i}+2\hat{j}+\hat{k}) = 9$ is:

(a) 2 units

(b) 3 units

(c) 4 units

(d) 5 units

(Ans: (a))

Q3. The location of two points $\hat{i}-\hat{j}+2\hat{k}$ and $2\hat{i}+3\hat{j}-\hat{k}$ with respect to plane $\vec{r}\cdot(\hat{i}-3\hat{j}+5\hat{k})=5$ is:

(a) one lies on the plane and other lies outside the plane

(b) both lies on the plane

(c) lies on opposite side of plane

(d) lies on same side of plane

(Ans: (c))

Q4. The intercepting form of plane $\vec{r}\cdot(\hat{i}+3\hat{j}-5\hat{k})=15$ is :

(a) $\vec{r}\cdot\left(\frac{1}{15}\hat{i}+\frac{1}{5}\hat{j}+\frac{1}{3}\hat{k}\right)=1$

(b) $\vec{r}\cdot\left(\frac{1}{15}\hat{i}+\frac{1}{5}\hat{j}-\frac{1}{3}\hat{k}\right)=1$

(c) $\vec{r}\cdot\left(\frac{1}{5}\hat{i}+\frac{1}{5}\hat{j}+\frac{1}{3}\hat{k}\right)=1$

(d) $\vec{r}\cdot\left(\frac{1}{5}\hat{i}+\frac{1}{5}\hat{j}+\frac{1}{5}\hat{k}\right)=1$

(Ans: (b))

Q5. The intercepts made by plane $\vec{r}\cdot(3\hat{i}+5\hat{j}+7\hat{k})=21$ is :

(a) $7, 7/5, 3$

(b) $7, \frac{3}{5}, 3$

(c) $7, \frac{21}{5}, 3$

(d) $7, \frac{22}{5}, 3$

(Ans: (c))

REFERENCES

1. सदिश विश्लेषण एवं सदिश कलन
द्वारा: डॉ॰ दीपक कुमार श्रीवास्तव
प्रकाशक: पियरसन, दिल्ली एवं चेन्नई, भारत