B.A./B.Sc. I (SEMESTER-II) - L.U.

MATHEMATICS - PAPER-II (GEOMETRY)

UNIT-II (PLANE - VECTOR APPROACH)

E-CONTENT / LECTURE NOTES-1

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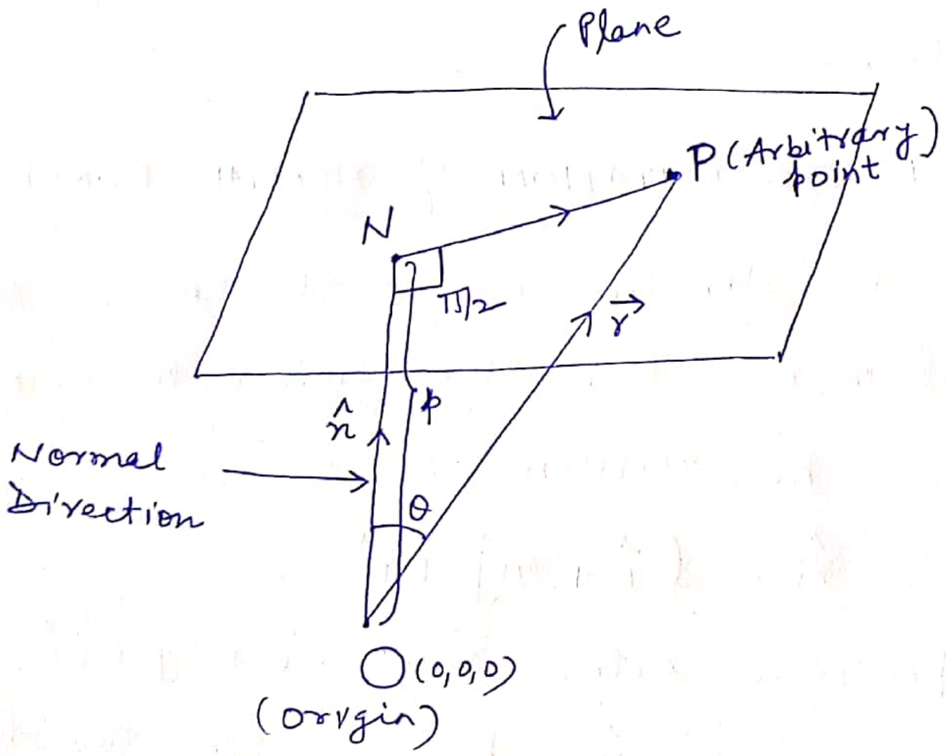
§ **DEFINITION.** Plane is a surface such that the line segment joining two points on the surface lies wholly on the surface.

§ **FORMS OF PLANE.** There are mainly two forms viz; Normal form and General form.

I. NORMAL FORM



Let 'p' be the length of perpendicular from origin O(0,0,0) onto the plane and $\hat{n}$ is the unit vector along this perpendicular ON. Then, the equation of plane is given by applying the fact that $\vec{ON}$ is perpendicular to $\vec{NP}$ where

$\vec{ON} = p\hat{n}$ and $\vec{NP} = \vec{OP} - \vec{ON} = \vec{r} - p\hat{n}$.

Now, by applying the perpendicularity condition between two vectors $\vec{ON}$ and $\vec{NP}$, we have

$$(p\hat{n})\cdot(\vec{r} - p\hat{n}) = 0$$

or $$\vec{r}\cdot\hat{n} - p(\hat{n}\cdot\hat{n}) = 0 \qquad (\because p \neq 0)$$

$\therefore$ $$\boxed{\vec{r}\cdot\hat{n} = p} \qquad (\because \hat{n}\cdot\hat{n} = 1)$$

This vector equation is known as NORMAL FORM of plane.

CARTESIAN REPRESENTATION of NORMAL FORM

Suppose the direction cosines of normal ON are l, m, n, then the unit vector along normal ON can be written as

$$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$$

and position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then from $\vec{r}\cdot\hat{n} = p$, we can have

$$(x\hat{i} + y\hat{j} + z\hat{k})\cdot(l\hat{i} + m\hat{j} + n\hat{k}) = p$$

$\Rightarrow$ $$lx + my + nz = p$$, the cartesian equation of NORMAL FORM of plane.

REMARK. Let 'θ' be the angle between OP and ON, then

$$ON = \text{the projection of } OP \text{ on } ON$$
$$= OP \cos\theta.$$

II. GENERAL FORM

Let $\vec{n}$ be any vector of magnitude $n = |\vec{n}|$ and having the same direction as that of $\hat{n}$.

Then by definition of unit vector

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{n}}{n} \quad \text{or} \quad \vec{n} = n\,\hat{n}.$$

Now, from normal form of plane (Vector-eqn),

we have $\quad \vec{r}\cdot\hat{n} = p$

or $\quad \vec{r}\cdot\left(\frac{\vec{n}}{n}\right) = p \Rightarrow \vec{r}\cdot\vec{n} = np$
$= q$ (say)

where $q = np \Rightarrow \boxed{p = ON = \frac{q}{n}}$

The vector equation $\boxed{\vec{r}\cdot\vec{n} = q}$ is the general equation of plane perpendicular to the direction represented by $\vec{n}$.

CARTESIAN REPRESENTATION of GENERAL FORM

Suppose the direction ratios of normal ON are a, b, c, then the vector along normal ON can be written as

$$\vec{ON} = \vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$$

and position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then from $\vec{r} \cdot \vec{n} = q$, we can have

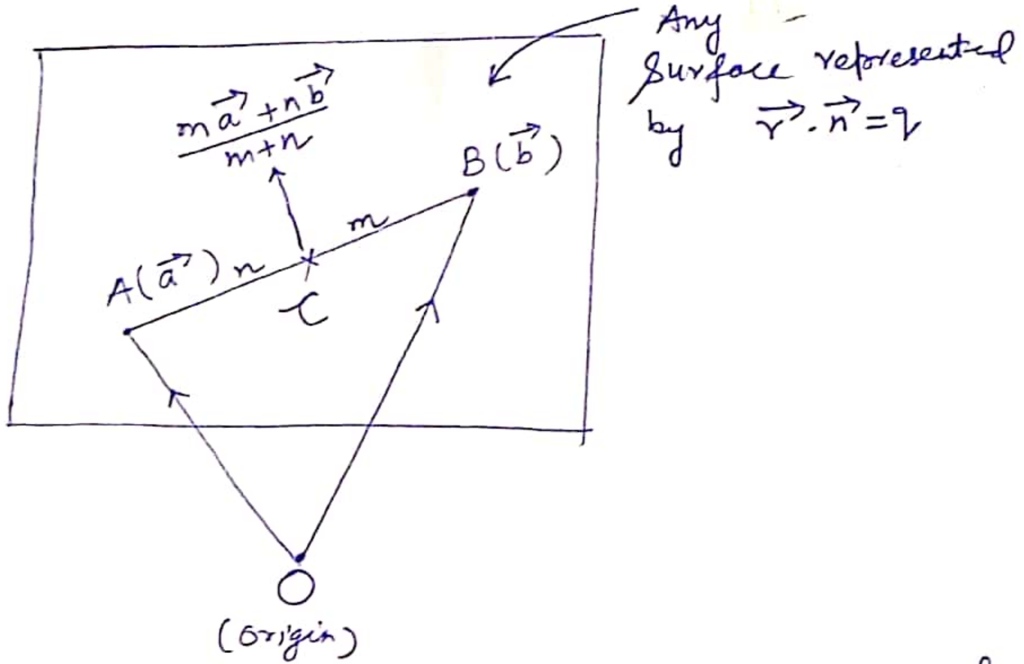
$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = q$$

$\Rightarrow$ $ax + by + cz = q$, the cartesian equation of GENERAL FORM of plane.

REMARK. If $ON = p = 0$, then the equation of plane $\vec{r} \cdot \hat{n} = 0$ or $\vec{r} \cdot \vec{n} = 0$ is the equation of plane passes through the origin O.

THEOREM. Vector equation $\vec{r} \cdot \vec{n} = q$ always represents a plane.

PROOF. Let $A(\vec{a})$ and $B(\vec{b})$ are any two points on the surface having position vectors $\vec{a}$ and $\vec{b}$ with respect to origin O.



As the points $A(\vec{a})$ and $B(\vec{b})$ are on surface given by equation $\vec{r}\cdot\vec{n}=q$, then, we get

$$\vec{a}\cdot\vec{n}=q \quad \text{— (1)} \quad \text{and} \quad \vec{b}\cdot\vec{n}=q \quad \text{— (2)}$$

Now, multiplying (1) by m and (2) by n, on adding, we get

$$(m\vec{a}+n\vec{b})\cdot\vec{n}=(m+n)q$$

$$\Rightarrow \left(\frac{m\vec{a}+n\vec{b}}{m+n}\right)\cdot\vec{n}=q \qquad (A)$$

$\therefore$ Equation (A) shows that the point whose position vector $\frac{m\vec{a}+n\vec{b}}{m+n}$ also lies on the surface $\vec{r}\cdot\vec{n}=q$.

But $\frac{m\vec{a}+n\vec{b}}{m+n}$ is any point C on the line segment joining $A(\vec{a})$ and $B(\vec{b})$ which divides AB in ratio $m:n$. Thus, every point on the line joining A & B satisfies the equation $\vec{r}\cdot\vec{n}=q$, which is the definition of plane.

Hence, the surface having equation $\vec{r}\cdot\vec{n}=q$ always represents a plane. **PROVED.**

SOLVED EXAMPLES BASED ON LECTURE-1

EXAMPLE-1 The perpendicular length from origin $O(0,0,0)$ on to the plane $\vec{r}\cdot(6\hat{i}+3\hat{j}+2\hat{k}) = 49$ is:

(a) 6 units

(b) 7 units

(c) 8 units

(d) 9 units

Solution. The perpendicular length ON from origin $O(0,0,0)$ onto the plane $\vec{r}\cdot\vec{n} = q$ is $\frac{q}{n}$ where $n = |\vec{n}|$.

Here, $\vec{n} = 6\hat{i}+3\hat{j}+2\hat{k}$, $|\vec{n}| = \sqrt{(6)^2+(3)^2+(2)^2}$

$= \sqrt{36+9+4}$

$= \sqrt{49} = 7$.

So, $ON = \frac{q}{n} = \frac{49}{7} = 7$ units.

$\therefore$ Answer is (b).

EXAMPLE-2 The normal form of plane (General form) $\vec{r}\cdot(6\hat{i}+3\hat{j}+2\hat{k}) = 35$ is:

(a) $\vec{r}\cdot(-\frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{2}{7}\hat{k}) = 5$

(b) $\vec{r}\cdot(\frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{2}{7}\hat{k}) = 5$

(c) $\vec{r}\cdot(\frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{2}{7}\hat{k}) = 5$

(d) $\vec{r}\cdot(\frac{6}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{2}{7}\hat{k}) = 5$

Solution. We know that the normal form of plane (General form) $\vec{r}\cdot\underbrace{(6\hat{i}+3\hat{j}+2\hat{k})}_{=\vec{n}} = 35 = q$.

is $\vec{r}\cdot\hat{n} = p$ where $q = np$

& $ON = p$, $n = |\vec{n}| = 7$

So, the normal form of given plane is

$$\vec{r}\cdot(6\hat{i}+3\hat{j}+2\hat{k}) = 35$$

divide this vector equation by $n = |\vec{n}| = 7$, then we have $\vec{r}\cdot\left(\frac{6\hat{i}+3\hat{j}+2\hat{k}}{7}\right) = \frac{35}{7} = 5$

or $\vec{r}\cdot\left(\frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{2}{7}\hat{k}\right) = 5$ which is

in the form $\vec{r}\cdot\hat{n} = p$ where $\hat{n} = \frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{2}{7}\hat{k}$

and $p = ON = 5$ units.

So, the answer is (b).

EXAMPLE-3 Plane passes through the origin O is:

(a) $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 0$

(b) $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 1$

(c) $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = -1$

(d) $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 2$

Solution. We know that equation of plane passes through the origin O is $\boxed{\vec{r}\cdot\vec{n} = 0}$.

So, the answer is (a).

EXAMPLE - 4 The cartesian form of plane $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 5$ is:

(a) $x+2y-3z=5$

(b) $x-2y+3z=5$

(c) $-x+2y+3z=5$

(d) $x+2y+3z=5$

Solution. Take the position vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then equation of plane becomes

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 5$$

i.e, $$x+2y+3z=5.$$

So, the answer is (d).

EXAMPLE - 5 The cartesian form of Normal equation of plane $\vec{r}\cdot(6\hat{i}+3\hat{j}+2\hat{k}) = 35$ is:

(a) $\frac{6}{7}x+\frac{3}{7}y+\frac{2}{7}z=8$

(b) $\frac{6}{7}x+\frac{3}{7}y+\frac{2}{7}z=7$

(c) $\frac{6}{7}x+\frac{3}{7}y+\frac{2}{7}z=5$

(d) $\frac{6}{7}x+\frac{3}{7}y+\frac{2}{7}z=6$

Solution. The normal equation of plane (see Example-2) is $\vec{r}\cdot\left(\frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{2}{7}\hat{k}\right)=5$

Take position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then Cartesian equation of Normal form becomes

$$\frac{6}{7}x + \frac{3}{7}y + \frac{2}{7}z = 5.$$

So, the answer is (c).

EXAMPLE-6 The direction cosines of normal drawn from origin O onto the plane $\vec{r}\cdot(6\hat{i} + 3\hat{j} + 2\hat{k}) = 35$ are:

(a) $\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$

(b) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

(c) $\frac{6}{7}, \frac{3}{7}, -\frac{2}{7}$

(d) $-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$

Solution. The normal form of given plane (see Example-2) is

$$\vec{r}\cdot\left(\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right) = 5$$

or $\vec{r}\cdot\hat{n} = 5 \ (=p)$

where $\hat{n} = \frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}$ is the unit vector along the normal ($\perp r$) drawn from origin O onto the given plane. So, the d.c.'s of normal are $\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$.

So, the answer is (a).

EXERCISE

(BASED ON LECTURE-1)

Q1. The perpendicular length from origin $O(0,0,0)$ on to the plane $\vec{r}\cdot(2\hat{i}+2\hat{j}+\hat{k})=15$ is:

(a) 2 units

(b) 3 units

(c) 4 units

(d) 5 units

(Ans: d)

Q2. The normal form of plane (General form) $\vec{r}\cdot(2\hat{i}+2\hat{j}+\hat{k})=15$ is :

(a) $\vec{r}\cdot\left(-\frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}\right)=5$

(b) $\vec{r}\cdot\left(\frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}\right)=5$

(c) $\vec{r}\cdot\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}\right)=5$

(d) $\vec{r}\cdot\left(\frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}\right)=5$

(Ans: (b))

Q3. The plane passes through origin $O(0,0,0)$ is:

(a) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$

(b) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=0$

(c) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=-1$

(d) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$

(Ans: (b))

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Q 4. The cartesian form of Normal equation of plane $\vec{r}\cdot(2\hat{i}+2\hat{j}+\hat{k})=15$ is :

(a) $\frac{2}{3}x+\frac{2}{3}y-\frac{1}{3}z=5$

(b) $\frac{2}{3}x-\frac{2}{3}y+\frac{1}{3}z=5$

(c) $\frac{2}{3}x+\frac{2}{3}y+\frac{1}{3}z=5$

(d) $-\frac{2}{3}x+\frac{2}{3}y+\frac{1}{3}z=5$

(Ans: (c))

Q 5. The direction cosines of normal drawn from origin O onto the plane $\vec{r}\cdot(2\hat{i}+2\hat{j}+\hat{k})=15$ are :

(a) $-\frac{2}{3},\frac{2}{3},\frac{1}{3}$

(b) $\frac{2}{3},-\frac{2}{3},\frac{1}{3}$

(c) $\frac{2}{3},\frac{2}{3},-\frac{1}{3}$

(d) $\frac{2}{3},\frac{2}{3},\frac{1}{3}$

(Ans: (d))

Q 6. If 1, 3, 5 are the d.c.'s of normal (ON) drawn from origin O onto the plane and Length ON = 7 units, then normal equation of plane is:

(a) $\vec{r}\cdot(\hat{i}+3\hat{j}+5\hat{k})=7$

(b) $\vec{r}\cdot(-\hat{i}+3\hat{j}+5\hat{k})=7$

(c) $\vec{r}\cdot(\hat{i}-3\hat{j}+5\hat{k})=7$

(d) $\vec{r}\cdot(\hat{i}+3\hat{j}-5\hat{k})=7$

(Ans: (a))

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